AD NUMBER

AD231936

NEW LIMITATION CHANGE

TO

Approved for public release, distribution unlimited

FROM

Distribution authorized to DoD only; Administrative/Operational Use; 10 NOV 1959. Other requests shall be referred to Office of Naval Research, 875 North Randolph Street, Arlington, VA 22203. Pre-dates formal DoD distribution statements. Treat as DoD only.

AUTHORITY

ONR ltr dtd 15 Jun 1977

UNCLASSIFIED

AD 231 936

Reproduced
by the
Armed Services Technical Information Agency

ARLINGTON HALL STATION; ARLINGTON 12 VIRGINIA

UNCLASSIFIED

Minimum Variance Unbiased Estimates

Generalization of Thompson's Distribution

Random Orthonormal Bases

By

Andre G. Laurent

TECHNICAL REPORT NO. 5

November 10, 1959

CONTRACT NONR - 2575(00)
(NR-042-201)
for
Office of Naval Research

DEPARTMENT OF MATHEMATICS
WAYNE STATE UNIVERSITY
Detroit 2, Michigan

Minimum Variance Unbiased Estimates.

Generalization of Thompson's Distribution.

Random Orthonormal Bases.

by Andre G. Laurent
Department of Mathematics
Wayne State University

1.0 Let X_1---X_N be a sample of N independent observations with a normal distribution $N(m,\sigma)$. In 1935, W. R. Thompson has obtained the probability distribution of an observation X_i chosen at random in the sample, when the mean m and the standard deviation σ are replaced by the corresponding sample characteristics [1]; namely $(X_i-\bar{X})^2/(N-1)S^2$ follows an incomplete beta distribution with parameters 1/2 and (N/2)-1.

This result has been generalized in several directions, in [2]; the distribution of a sample is given after its being centered and studentized by means of the mean and standard deviation of an independent sample; the distribution of a subsample is derived when m and σ are replaced by the corresponding sample characteristics, and the conditional distribution of a subsample is obtained given $\bar{X}$ and S.

Further generalizations have been presented in [3] where the results just mentioned are extended to the case of the multivariate normal distribution. Applications to Bombing Theory were proposed in [4].

1.1 Let $(\xi) = (\xi_1, \ldots, \xi_i \ldots, \xi_k)'$ with $\xi_i = (X_i^1, \ldots, X_i^2, \ldots X_i^p)$ be a subset of observations from a set of N independent observations $(X) = (X_1, \ldots, X_i, \ldots X_N)' = (X_i^j)$ with the same p-variate normal distribution $N(m, \Sigma)$; let m' and S be the vector mean and covariance matrix of the subset (ξ). The <u>distribution of ξ given m' and S for $k < N-p$ is</u>

$$f(\xi \mid m'S)\,d\xi = C.\left| I - (k/N)S^{-1}S_\xi - [k/(N-k)]S^{-1}S^* \right|^{(N-k-p-2)/2} d\xi / S^{k/2}$$

in the domain where the determinants are positive, with

$$S^* = (m'_\xi - m')'(m'_\xi - m')$$

and

$$C = \frac{\Gamma\text{[illegible]} \ldots \Gamma\text{[illegible]} / \Gamma[(N-k-1)/2] \ldots \Gamma[(N-k-p)/2] \ldots}{\pi^{kp/2} N^{(k-1)p/2} (N-k)^{p/2}}$$

This result was given in [3].

2.1 In view of the completeness of m'S, it results from Blackwell's theorem that

$$\int \ldots \int \varphi(\xi) f(\xi \mid m'S)\,d\xi$$

will provide a uniformly minimum variance unbiased estimate of $E[\varphi(\xi)]$ if it exists.

As a special application, if

$$g(A, m, \Sigma) = \int_A \ldots \int N(m, \Sigma)\,dX,$$

a minimum variance unbiased estimate of $g(A, m, \Sigma)$ will be given by $\int_A \ldots \int \hat{f}(\xi \mid m'S)\,d\xi$ with $k = 1$, since an unbiased estimate of $g(A, m, \Sigma)$ is provided by the characteristic function $1_A(\xi)$ of the set A whose conditional expected value is given by the integral above. Further, in view of the fact that summation keeps unbiasedness, the minimum variance unbiased estimate of the density $N(m, \Sigma)$ is $f(\xi \mid m'S)$.

The already established results for the univariate distribution can be obtained by considering it as a special case of the multivariate situation.

2.2 The conditional probability distribution of m'_ξ, S_ξ given m', S is obtained by writing the joint distribution of the mean m'_n and covariance matrix S_n of a sample of size $n = N-k$ and of m'_ξ and S_ξ, then performing the change of variable

$$m'_n = Nm'/n - km'_\xi/n$$
$$S_n = NS/n - kS_\xi/n - S^*kN/n^2$$

subsequently dividing by the distribution of S.

m'_n is $N(m, \Sigma/n)$ distributed, S_n is $W(\Sigma, n-1)$ distributed, the Jacobian of the change of variables is

$$\frac{D(m'_n, Sn)}{D(m', S)} = (N/n)^{p(p+3)/2}$$

m' is $N(m, \Sigma/N($, S is $W(\Sigma, N-1)$ distributed.

After grouping terms, oneobtains for $p < k < N-p$

$$f(m'_\xi, S_\xi \mid m', S)\,dm'_\xi\, dS_\xi =$$
$$C\left|I-(k/N)S_\xi S^{-1} - [k/(N-k)]S^*S^{-1}\right|^{(N-k-p-2)/2}\left|S_\xi S^{-1}\right|^{(k-p-2)/2}$$
$$(dm'_\xi / |S|^{1/2})\; dS_\xi / |S|^{(p+1)/2}$$

where

$$C = \left[\frac{k^k}{(N-k)N^{k-1}}\right]^{p/2}[1/\pi^{p(p+1)/4}]\prod_1^p \frac{\Gamma[N-j)/2]}{\Gamma[N-k-j)/2]\Gamma[(k-j)/2]}$$

in the domain where the determinants are positive.

2.3 In view of the completeness of m', S, the kernel $f(m'_\xi, S_\xi \mid m', S)$ is of great interest

If $\varphi_n(m'_n, S_n)$ is an unbiased estimate of a function $g(m, \Sigma)$ of the parameters, then, as a consequence of the uniqueness of

minimum variance unbiased linear estimates in case of completeness

$$\varphi_N(m', S) = \int \cdots \int \varphi_k(m'_\xi, S_\xi) f(m'_\xi, S_\xi \mid m', S)\, dm'_\xi \, dS_\xi$$

i.e. minimum variance unbiased linear estimates are a subclass of the solutions of this integral equation which defines a multivariate transformation.

3.0 In case X is normally distributed $N(0,\sigma)$, univariate situation, or $N(0,\Sigma)$, multivariate situation, other generalizations are of interest.

3.1 In the univariate case, let X be a set of N observations $X_1 \ldots X_N$ with probability distribution $N(0,\sigma)$, the conditional distribution of a subsample ξ of k items drawn from X at random, given the moment of order two $m'_2(X)$ of the set X, is obtained by writing the joint distribution of the moment of order two, $m'_2(X_n)$ of a sample X_n of size $n = N-k$ and of an independent sample ξ of k items, performing the change of variables $m'_2(X_n) = (N/n)m'_2(X) - (k/n)m'_2(\xi)$ whose Jacobian is N/n, then dividing by the distribution of $m'_2(X)$, one obtains with $\xi = (X_1 \ldots X_k)$, $k < N$

$$f[\xi \mid m'_2(X)]d\xi = \frac{\Gamma(N/2)}{\pi^{k/2}\Gamma[(N-k)/2]} \left[1-(k/N)\frac{m'_2(\xi)}{m'_2(X)}\right]^{(N-k-2)/2} \frac{d\xi}{[Nm'_2(X)]^{k/2}}$$

in the domain where the bracket is positive.

If $\tau = \xi / \sqrt{m'_2(X)}$, one obtains

$$\frac{\Gamma(N/2)}{\pi^{k/2}\Gamma[(N-k)/2]}\left(1-\frac{\zeta\zeta'}{N}\right)^{(N-k-2)/2}(d\zeta/N^{k/2}) \qquad \zeta\zeta' \leq N$$

i.e. the distribution of a subsample normalized by $\sqrt{m_2'(X)}$ is dependent from m_2'.

Also, in case $k = N-1$, $u = \xi/\sqrt{Nm_2'(X)}$, $u_1 \ldots u_{N-1}$ are $N-1$ independent coordinates of a N dimensional unit vector, and one has
$[\Gamma(N/2)/\pi^{N/2}](1 - uu')^{-1/2}du \qquad uu' \leq 1.$

The distribution of any coordinate u_i being

$$\frac{1}{B\left(\frac{N-1}{2}, \frac{1}{2}\right)}(1-u_i^2)^{(N-3)/2}du_i \qquad u_i^2 \leq 1$$

(if $N = 3$ one obtains a rectangular distribution).

Clearly, the quantity $v = \sqrt{N-1}\, u_i/\sqrt{1-u_i^2}$ is Student distributed with $N-1$ degrees of freedom.

u_i is the cosine of the angle φ_i of the unit vector with any direction, $u_i = \cos\varphi_i$ and $v = \sqrt{n-1}\,\cot.\,\varphi_i$.

3.2 This result can be slightly generalized to the case of a spherical distribution.

Let $X = (X_1, \ldots, X_n)$ have probability distribution $f(X)dX$, let $R^2 = XX'$, let $f(X)dX$ be spherical, i.e. $f(X) = h(XX')$, then R^2 is a sufficient statistic for $f(X)dX$
$f(X)dX = h(R^2)dX = h(R^2)R^{n-1}dRd\Sigma_{n,1}$ ($\Sigma_{n,1}$ = the area of the unit sphere in the n dimensional space); integrating out $d\Sigma_{n,1}$, one obtains $\Sigma_{n,1}h(R^2)R^{n-1}dR$ as the probability distribution of R, a well known result.

More explicitly, using polar coordinates

$X_1 = R\cos\varphi_1$

$X_{n-1} = R\sin\varphi_1 \sin\varphi_2 \ldots \sin\varphi_{n-2}\cos\varphi_{n-1}$

$X_n = R\sin\varphi_1 \ldots\ldots \sin\varphi_{n-2}\sin\varphi_{n-1}$

φ_i in (o,π) if $i < n-1$, φ_i in$(o,2\pi)$ if $i = n-1$

$dX = J dR d\varphi_1 \ldots d\varphi_{n-1}$, where

$J = [D(X)/D(R,\varphi)] = R^{n-1}\sin^{n-2}\varphi_1 \ldots \sin\varphi_{n-2}$

and $d\Sigma_{n,1} = \sin^{n-2}\varphi_1 \ldots \sin\varphi_{n-2} d\varphi_1 \ldots d\varphi_{n-1} = J d\varphi / R^{n-1}$

$= dX_1 \ldots dX_{n-1} / |X_n| R^{n-2}$

hence, the conditional distribution of X given R^2 is

$$\frac{d\Sigma_{n,1}}{\Sigma_{n,1}} = \frac{\Gamma(n/2)}{2\pi^{n/2}} \sin^{n-2}\varphi_1 \ldots \sin\varphi_{n-2} d\varphi_1 \ldots d\varphi_{n-1}$$

it is the distribution of the independent polar coordinates of a unit vector equidistributed on the unit sphere. In terms of the X_i's and u_i's, this gives (taking into account both signs of X_n),

$$\frac{\Gamma(n/2)}{\pi^{n/2}} \frac{dX_1 \ldots dX_{n-1}}{|X_n| R^{n-2}} = \frac{\Gamma(n/2)}{\pi^{n/2}} (I-uu')^{-1/2} du$$

Consequently the distribution of any subsample is the one given in paragraph 3.1.

This shows that "Thompson's distribution depends only on the sphericity of the universe. If we consider a random unit vector, with fixed origin, "chosen at random" in the n space i.e., with a probability distribution invariant under the rotation group, we see that we can construct such a vector by normalizing a vector whose distribution is spherical.

3.4 Let us consider a set (X) of independent N observations of a p-dimensional vector with distribution $N(o, \Sigma)$. $(X) = (X_1, \dots, X_i, \dots, X_N)' = (X_i^j) = (X^1, \dots, X^j \dots X^p), X^j = (X_1^j \dots X_i^j \dots X_N^j)'$ let (ξ) be a subset of K observations. Let S and S_ξ be the maximum likelihood estimates of Σ obtained respectively with (X) and (ξ). The distribution of (ξ) given the sufficient statistic S, is obtained by writing the joint distribution of (ξ) and the maximum likelihood estimate S_n of Σ obtained with a sample of $n = N-k$ observations, making the change of variables

$S_n = (N/n)\, S - (k/n) S_\xi$

and dividing by the distribution of S. One obtains

$$f(\xi \mid S)\, d\xi = C.\, \left| I - (k/N) S^{-1} S_\xi \right|^{(N-k-p-1)/2} d\xi \,/\, |S|^{k/2}$$

with

$$C = \frac{\prod_1^p \Gamma[(N+1-j)/2]}{\prod_1^p \Gamma[(N-k+1-j)/2]} \, [1/(N\pi)^{pk/2}]$$

valid for $p \leq N-k$, in the domain where the determinants are positive.

Now $kS_\xi = \xi'\xi = \xi'_1 \xi_1 + \dots + \xi'_k \xi_k$

so that

$$\left| I - (k/N) S^{-1} S_\xi \right| = \left| I - (S^{-1}/N) \sum_1^k \xi'_j \xi_j \right| = |NS|^{-1} \left| S - \sum^k \xi'_j \xi_j \right|$$

there exists a unique triangular matrix T with positive diagonal such that $NS = X'X = TT'$ then

$$\left| I - (k/N) S^{-1} S_\xi \right| = \left| T^{-1} \right| \left| S - \sum \xi'_j \xi_j \right| \left| T'^{-1} \right| =$$

$$\left| I - \sum T^{-1} \xi'_j \xi_j T'^{-1} \right| = \left| I - \sum \eta'_j \eta_j \right| \quad \text{if } \eta'_j = T^{-1} \xi'_j \quad \eta_j = \xi_j T'^{-1}$$

Now, it is known that

$$\left| I - \sum \eta_j' \eta_j \right| = \left| \vartheta_{il} - \eta_i \eta_l' \right|$$

so that

$$\left| I - (k/N) S^{-1} S_\xi \right| = \left| \vartheta_{il} - \xi_i T'^{-1} T^{-1} \xi_l' \right| = \left| \vartheta_{il} - \xi_i (S^{-1}/N) \xi_l' \right|$$

$$f(\xi \mid S) d\xi = C. \left| \vartheta_{il} - \xi_i (S^{-1}/N) \xi_l' \right|^{(N-k-p-1)/2} d\xi \,/\, \left| S \right|^{k/2}$$

One can also work with the variables η making the transformation $\eta = \xi T'^{-1}$, one obtains for the distribution of η,

$$C! \left| I - \eta' \eta \right|^{(N-k-p-1)/2} d\eta$$

since the Jacobian $D(\xi)/D(\eta) = \left| T \right|^k = \left| S \right|^{k/2}$,

$\eta = \xi T'^{-1}$

is a procedure of studentization of the set ξ and generalizes the studentization by $R = \sqrt{N m_2'(X)}$ in the univariate case to the multivariate case.

In some problems the following remark is of interest.

Consider theorthogonal matrix O that diagonalizes (S^{-1}/N) into Λ^{-1} say, then premultiplying and postmultiplying by O^{-1} and O we have

$$\left| I - (k/N) S^{-1} S_\xi \right| = \left| I - \Lambda^{-1} \sum^k O^{-1} \xi_j' \xi_j O \right| = \left| I - \Lambda^{-1} \sum^k u_i' u_i \right|$$

where u_i is the vector of the coordinate of ξ_i w.r. to the system of coordinates constituted by the eigen vectors of (S^{-1}/N) i.e. of NS, (which is almost surely of rank p).

One can consider the normalization of these coordinates by the roots $\sqrt{h_1} .. \sqrt{h_p}$ of $\Lambda^{1/2}$, we get, in terms of the new variables V

$$C\left|I - V'V\right|^{(N-k-p-1)/2} dV$$

and the corresponding normalization is

$$V = \xi S^{-1/2}$$

according to the usual definition of $S^{1/2}$ (see, for instance, [13]).

3.5 The conditional distribution of S_ξ given S is obtained by writing the joint distribution of S_ξ and S_n, making the change of variables indicated in 3.4 and dividing by the distribution of S, one obtains

$$f(S_\xi \mid S)dS = C. \left|I - (k/N)S_\xi S^{-1}\right|^{(N-k-p-1)/2} \left|S_\xi S^{-1}\right|^{(k-p-1)/2} dS_\xi \Big/ \left|S\right|^{(p+1)/2}$$

with

$$C = (k/N)^{pk/2} \frac{1}{(\pi)^{p(p-1)/4}} \frac{\prod_1^p \Gamma[(N+1-j)/2]}{\prod_1^p \Gamma[(k+1-j)/2] \prod_1^p \Gamma[(N-k+1-j)/2]}$$

$p \leq k \leq N-p$, in the domain where the determinants are positive.

In terms of η one obtains

$$C \left|I - \eta'\eta\right|^{(N-k-p-1)/2} \left|\eta'\eta\right|^{(k-p-1)/2} d\eta'\eta$$

since the Jacobian of the transformations from $\xi'\xi = kS_\xi$ to $\eta'\eta$ is $\left|T\right|^{p+1} = \left|S\right|^{(p+1)/2}$. This is a generalization of the incomplete beta distribution.

3.6 The results in 3.4 and 3.5 can be generalized to the case of a probability distribution which depends on the set X through $X'X = NS$, i.e.

$$f(X)dX = h(X'X)dX$$

Then $X'X$ is a sufficient statistic for the distribution. It has been shown by Hsu that the probability distribution of $X'X = NS$, under such circumstances, is

$$g(X'X)dX'X = \frac{\pi^{Np/2 - p(p-1)/4}}{\prod_{1}^{p}\Gamma[(N-j+1)/2]}\left|X'X\right|^{(N-p-1)/2} h(X'X)dX'X \quad [9]$$

It is well known that there exists a unique triangular matrix T with positive diagonal such that $X'X = TT'$ andthat the transformation $X = YT'$ defines the socalled "rectangular coordinates" T [8]. To a considerable extent, what follows overlaps the theory of rectangular coordinates, though the approach is somewhat different.

3.6.1 The vectors $(X^1,\ldots,X^j,\ldots X^p,)$, span a space that is almost surely p-dimensional. We want to construct an orthonormal basis $(Y^1,\ldots,Y^j\ldots Y^p)$ in that space. This can be done by use of the Schmidt's orthogonalization process, starting with X^1.

$(X^1,\ldots,X^j,\ldots X^p) = U(Y^1,\ldots,Y^j,\ldots Y^p)$

where U is an upper triangular operator (i.e. in the system $(Y^1,\ldots,Y^p)$, u is represented by an upper triangular matrix with positive diagonal.)

now $X'X = NS = ((X^i X^j)) = ((UY^i UY^j)) = ((Y^i U'UY^j))$

i.e. the i,j element of $U'U$ matrix of $U'U$ with respect to $(Y^1,\ldots,Y^p)$ is the i,j element of NS that is $U'U = TT'$ and from the uniqueness of the factorization $U = T'$; consequently

$X = YT'$

Y has orthogonal columns and $Y'Y = I_{pp}$.

$X_j = t_{j1}Y^1 +\ldots+t_{jj}Y^j$

X involves np free random variables, Y, np−p(p+1)/2 as orthogonal and T', p(p+1)/2 as triangular,

$Y = X\,T'^{-1}$ is a random orthonormal basis.

Now

$$\rho(X)\,dX = h(X'X)\prod_j dX^j_1 \ldots dX^j_N$$

let us study

$$\prod_j dX^j_1 \ldots dX^j_N = \prod_j dX^j$$

One has $X^j = (t_{j1}Y^1 + \ldots + t_{jj-1}Y^{j-1}) + t_{jj}Y^j$

Consider the j−1 dimensional space spanned by $Y^1 \ldots Y^{j-1}$ and complete that basis by an orthonormal basis in the N−j+1 dimensional complementary space; $t_{jj}Y^j$ is the projection of X^j in that space where its coordinates are $X^{j*}_j, \ldots, X^{j*}_N$ say. In the new N dimensional basis X^j has coordinates $t_{j,1} \ldots t_{j,j-1} X^{j*}_j \ldots X^{j*}_N$ and since one passes from the old basis to the new basis by a rotation one has

$$dX^j = dt_{j,1} \ldots dt_{j,j-1}\, dX^{j*}_j \ldots dX^{j*}_N$$

Using polar coordinates in the N−j+1 complementary space we have

$$dX^{j*}_j \ldots dX^{j*}_N = t^{N-j}_{jj} \quad dt_{jj}\, d\Sigma_{N-j+1}$$

where $d\Sigma_{N-j+1}$ denotes the elementary area of the unit sphere in the N−j+1 dimensional space.

Therefore,

$$dX = dT' \prod_j t^{N-j}_{jj} \; d\Sigma_{N-j+1}$$

and since $dT' = 2^{-p}\prod_j t^{-p+j-1}_{jj}\, dX'X$ and $\prod_j t_{jj} = \left| T \right|$, one has

$$dX = 2^{-p}\left| X'X \right|^{(N-p-1)/2} \prod_j d\Sigma_{N-j+1}\, dX'X$$

$d\Sigma_{N-j+1}$ symbolizes a differential expression involving N-j angles functionally independent [i.e. Np-p(p+1) variables for all j from 1 to p.]

$$f(X)dX = h(X'X)2^{-p}\left|X'X\right|^{(N-p-1)/2}\prod_j d\Sigma_{N-j+1}dX'X$$

integrating the angles out, one obtains g(X'X) dX'X, mentioned above, so that the conditional distribution of X given X'X is

$$f(X|X'X)dX = \prod_j \frac{d\Sigma_{N-j+1}}{\Sigma_{N-j+1}}$$

when expressed in function of the angular coordinates of the polar systems of coordinates.

$$\frac{d\Sigma_{N-j+1}}{\Sigma_{N-j+1}}$$

is the distribution of a unit vector Y^j equidistributed on the unit sphere of the N-j+1 dimensional space.

The distribution of X given X'X is independent from the nature of f, therefore, all the results of 3.4 and 3.5 are valid without the assumption of normality, it is enough that $f(X) = h(X'X)$. Making the change of variables X = YT' and integrating out T', we obtain the distribution above, (since T and X'X are equivalent as sufficient statistics), i.e. the Y have the distribution $\prod_j d\Sigma_{N-j+1}/\Sigma_{N-j+1}$; more specifically, the random basis $(Y^1,\ldots,Y^p)$ is constituted with unit vectors Y^j that are uniformly distributed, given $Y^1\ldots Y^{j-1}$, on theunit sphere of the N-j+1 dimensional space and the distribution of Y is the same as that of X given X'X = I.

$Y = XT'^{-1}$ can be considered as a studentization or normalization of X generalizing the normalization by $\sqrt{Nm_2'(X)}$ in the univariate situation. One can write

$$\prod_j \frac{d\Sigma_{N-j+1}}{\Sigma_{N-j+1}} = \prod_j \frac{\Gamma[(N-j+1)/2]d\Sigma_{N-j+1}}{2^p\,\pi^{Np/2-p(p-1)/4}}$$

Y^j has coordinates $Y_j^{j*} \ldots Y_N^{j*}$ with respect to any orthonormal basis spanning the N-j+1 space it belongs to, and the distribution of those is

$$\frac{\Gamma[(N-j+1)/2]dY_j^{j*}\ \ldots\ dY_{N-1}^{*j}}{\pi^{(N-j+1)/2}[1-(Y_j^{j*2}+\ldots+Y_{N-1}^{j*2})]^{1/2}}$$

$$= \frac{2\Gamma[(N-j+1)/2)}{\pi^{(N-j+1)/2}}\ \frac{dY_j^{j*}\ldots dY_{N-1}^{j*}}{\left|Y_N^{j*}\right|}$$

(Y_N^j can take <u>two</u> values when the other dependent coordinates of Y^j are fixed.)

3.6.2 To obtain the distribution of X given X'X as an explicit function of the X_i^j or the Y_i^j, one ought to pass from the Y* to the Y, choosing Np-p(p+1) functionally independent Y_i^j, then, if needed, come back to X, through $X = YT'$; only NP-p(p+1)/2 X_i^j are functionally independent, given X'X. As a matter of fact, only N-j coordinates of X^j are independent, given X'X; this, in view of the fact that T is triangular and that only N-j coordinates of Y^j are independent. This shows that it is possible to get the

distribution of a subsample of k observations ξ, given $X'X$, only if the minimum of $N-j$, namely $N-p$, is at most equal to k. As an explicit function of the independent Y_i^j (that we will denote as Y^*) the probability distribution of Y, i.e. of Y^* is

$$\frac{2^p D(\Sigma_N, \ldots, \Sigma_{N-p+1})}{D(Y^*)} \; \frac{\prod dY_i^j}{\prod \Sigma_{N-j+1}}$$

To obtain the Jacobian, one can refer to [10] where it is shown that the Jacobian $D(X)/D(T,Y^*)$ of the change of variables $X = YT'$ is given by

$$\frac{D(X)}{D(T,Y^*)} = \frac{D(X,Y'Y)}{D(T,Y)} \Big/ \frac{D(Y'Y)}{D(Y^{**})}$$

(where Y^{**} denotes the set of dependent Y_i^j)

Now, $\dfrac{D(X,Y'Y)}{D(T,Y)} = 2^p \prod_1^p t_{ii}^{n-1}$ see [10]

and $\dfrac{D(T)}{D(X'X)} = 2^{-p} \prod_1^p t_{ii}^{-p+i-1}$

so that

$$f(X')dX = h(X'X)\left|X'X\right|^{(N-p-1)/2} dX'X \, \frac{D(Y^{**})}{D(Y'Y)} \, dY^*$$

$$\frac{D(\Sigma_N, \ldots, \Sigma_{N-p+1})}{D(Y^*)} = \frac{D(Y^{**})}{D(Y'Y)}$$

the distribution of Y^* is

$$(2^p/\prod\Sigma_{N-j+1})\ dY^*/[D(Y'Y)/D(Y^{**})]$$

where the general element of $Y'Y$ is $\sum_1^N Y_h^i Y_h^j$ and of Y^{**} is Y_{N-k+1}^{ℓ} for $k \leq \ell$ and o for $k > \ell$.

3.6.3 In case N = p, one can also use Cayley's parametric representation of Y.

If $|I+Y| \neq o$, i.e. if Y is non-exceptional, then there exists a skew-symmetric matrix S such that

$$Y = (I+S)^{-1}(I-S) = (I-S)(I+S)^{-1} = (I-S)/(I+S)$$
$$S=(I-Y)(I+Y)^{-1} = (I+Y)^{-1}(I-Y) = (I-Y)/(I+Y),$$

S non-exceptional since skew symmetric.

If $|I+Y| = o$, i.e. if Y is exceptional, then it is known that if one defines a sequence $\{J_r\}$ of diagonal matrices

$(\pm 1, \ldots \pm 1)$

$J_1 = (1 1 \ldots 1)$

$J_2 = (-1, 1 1 \ldots 1)$

$J_3 = (-1,-1, 1 \ldots 1)$

in such a way that J_r differs from J_{r-1} by changing the sign of only one diagonal element and if M_r is that set of matrices Y such that

$|Y+I| = o$

$|J_2 Y+I| = o$

$|J_{r-1} Y+I| = o$

$|J_r Y + I| \neq o$

the set of exceptional matrices is $\bigcup_{r>1} M_r$ and Cayley's represen-

tation for such a matrix is

$$Y = J_r\,(I-S)/(I+S),\quad S = (I-J_rY)/(I+J_rY)$$

for some $r > 1$

Let us, then, partition the space of X into the sets X* and X** such that, respectively,

$X^* = Y^*T^{*\prime}$, Y^* non-exceptional

and $X^{**} = Y^{**}T'^{**}$, Y^{**} exceptional,

one has

$$f(X^*)dX^* = h(X^{*\prime}X^*)dX^*$$

let us make the change of variables

$$X^* = [(I-S^*)/(I+S^*)]T'^*$$

whose Jacobian, after Hsu [11] is

$$D(X^*)/D(T^*_,S^*) = \prod_1^p t_{ii}^{p-i}\, 2^{p(p-1)/2}\left|I+S^*\right|^{-(p-1)}$$

then

$$dX^* = 2^{-p}2^{p(p-1)/2}\left|X^{*\prime}X^*\right|^{-1/2}\left|I+S^*\right|^{-(p-1)} dX^{*\prime}X^*dS^*$$

hence, the distribution of S^* given $X^{*\prime}X^*$ is independent from $X^{*\prime}X^*$ and given by

$$K dS^*/\left|I+S^*\right|^{p-1} \quad \text{with } K^{-1} = \int_{M_1} dS^*/\left|I+S^*\right|^{p-1}$$

K is given in [12] as $2^{-(p-2)(p-1)/2}\prod_1^p \pi^{K/2}/\Gamma(K/2)$

In case Y is exceptional, one makes the transformation

$$X^{**} = J_r[(I-S^{**})/I+S^{**})]T'^{**} \text{ when } Y^{**} \in M_r$$

It results from [12] that M_2 has same measure as M_1 and M_r has measure zero if $r > 2$. Considering only the set M_2 we will have

$$dx^{**} = 2^{-p}2^{p(p-1)/2}\left|X'^{**}X^{**}\right|^{-1/2}\left|I+S^{**}\right|^{-(p-1)} dX'^{**}X^{**}dS^{**}$$

and the distribution of S^{**} given $X'^{**}X^{**}$ is the same as that of S^* given X'^*X^*.

Given the proper definition of $S=S^*$ on M_1, $S=S^{**}$ on M_2 we will have.

$$\rho(X)dX = 2^{-p}2^{p(p-1)/2}h(X'X)(X'X)^{-1/2}\left|I+S\right|^{-(p-1)}dX'XdS$$

and given $X'X$, S has distribution

$$[(2^{p(p-1)/2})/(\prod_{1}^{p}\Sigma_{p-j+1})]\ dS/\left|I+S\right|^{p-1} \quad \text{i.e.}$$

$$\pi^{-p(p+1)/4}\prod_{1}^{p}\Gamma[(p-j+1)/2]2^{(p-1)(p-2)/2-1}\left|I+S\right|^{-(p-1)}dS$$

3.7 Consider the space defined by all X given $X'X$, if G is a group operating transitively on $X|X'X$, there exists at most one probability measure on that space that is invariant under G.

If we consider the group G of orthogonal matrices $O_{n,n}$ operating on X, $X \to OX$, it is geometrically obvious that G operates transitively on $X|X'X$, on the other hand, it is straight forward that the conditional distribution of X given $X'X$ is invariant under G since $(OX)'(OX) = X'X$, therefore it is the unique distribution on $X|X'X$ invariant under G.

Suppose now we want to "pick at random" a p dimensional orthonormal basis in the N dimensional space; by this is meant choosing X such that $X'X = I$ and that the probability distribution of X be invariant under the orthogonal group; it sufficies to take X with probability distribution $\rho(X)dX = g(X'X)dX$ and normalize by $X=YT'$, Y will be the basis desired. To construct such a basis, one can take $X^1,\ldots,X^p$ and use the Schmidts' orthogonalization process. Alternatively, one sees that one can choose at random a unit vector Y^1 in the N dimensional space, then a random unit fector Y^2 in the space complement to Y^1, and so on; at the jth step, one takes a random unit vector Y^j in the space complement to that spanned by $Y^1\ldots Y^{j-1}$. By random unit vector is meant a vector that is equidistributed on the unit sphere of the N-j+1 dimensional space.

3.8 From 3.5 we see that, in case $f(X)dX = h(X'X)dX$, a minimum variance unbiased estimate of a function of the parameters, if it exists and except for a condition of completeness on S, will have to be found among the solutions of the integral equation

$$\varphi_N(S) = \int \cdots \int \varphi_k(S)\, f(S_\xi \mid S)\, dS$$

4.0 Further generalizations of Thompson's distributions can be tried in several directions.

4.1 One can try to obtain the conditional distribution of a sub-sample ξ of a sample X with probability distribution $f(X)dX$, given the mean $\bar{X}$ and the variance S^2 of the sample, in the general case, when $f(X)$ is not spherical. This is possible when one has the probability distribution of the couple of statistics $\bar{X}, S^2$ for a sample of any size; references [5], [6], [7] deal with the derivation of such a distribution.

One writes the joint distribution of $\xi = (\xi_1 \dots \xi_k)'$ and of the variance and mean $S_n^2, \bar{X}_n$ of an independent sample of size $n = N-k$, (where N is the size of X), makes the change of variables, (as shown in [2]),

$$\bar{X}_n = \bar{X}N/(N-k) - \bar{\xi}\, k/(N-k)$$

$$S_n^2 = S^2N/(N-k) - (\bar{\xi} - \bar{X})^2/k^2(N-k)^2 - (\xi - \bar{X})'(\xi - \bar{X})/(N-k)$$

whose Jacobian is $(N/N-k)^2$ and divides by the distribution of $\bar{X}$, S^2.

If $\gamma_n(\bar{X}_n, S_n^2)$ is the probability density of $\bar{X}_n, S_n^2$ one has

$$f(\xi \mid \overline{X}, s^2) = \frac{\gamma_n[\overline{X}_n(\overline{X}, \overline{\xi}), s_n^2(\overline{X}, s^2, \overline{\xi}, s_\xi^2)]}{\gamma_N(\overline{X}, s^2)}(N/N-k)^2 f(\xi)d\xi$$

over the proper domain (namely $s_n^2 > o$) and the recurrence formula

$$\gamma_N(\overline{X}, s^2) = \int .. \int \gamma_n[\overline{X}_n(\overline{X}, \overline{\xi}), s_n^2(\overline{X}, s^2, \overline{\xi}, s_\xi^2) f(\xi)(N/N-k)^2 d\xi$$

where the summation is performed over the proper domain.

This formula may be useful when γ_n can be obtained directly without too much difficulty for small n.

In case n=2 one will perform the rotation

$$Y_1 = (X_1-X_2)/\sqrt{2} \qquad X_1 = (Y_1+Y_2)/\sqrt{2}$$

$$Y_2 = (X_1+X_2)/\sqrt{2} = \sqrt{2}\overline{X} \qquad X_2 = (Y_2-Y_1)/\sqrt{2}$$

$\overline{X} = Y_2/\sqrt{2} \quad s^2 = Y_1^2/2.$

4.2 One can also be interested in the distribution of the studentized ξ, namely $t = (\xi - \overline{X})/s$.

One writes the joint distribution of $\xi = (\xi_1 .. \xi_k)'$ and of $s_n^2, \overline{X}_n$, makes the change of variables $t = (\xi - \overline{X}_n)/s_n$ and integrates out $\overline{X}_n, s_n$; the density of t is

$$h(t) = \iint \gamma_n(\overline{X}_n, s_n^2) f(\overline{X}_n + t s_n) s_n^k d\overline{X}_n ds_n$$

then one makes the change of variables

$$\begin{aligned} t &= (t + \overline{t}k/N-k)s/s_n \\ &= (t+\overline{t}k/N-k)/[N/(N-k)-\overline{t}^2k^2/(N-k)^2-t't/(N-k)]^{1/2} \\ &= [t + \overline{t}k/(N-k)]/\{(N/N-k)[1 - \overline{t}^2k/(N-k) - s_t^2k/N]\}^{1/2} \end{aligned}$$

The Jacobian of the transformation is

$$\frac{D(t)}{D(\bar{t})} = [(N-k)/N]^{(k-2)/2} \left\{1 - [\bar{T}^2k^2/(N-k) + t't]/N\right\}^{-(k+2)/2}$$

$$= [(N-k)/N]^{(k-2)/2} \, [1 - \bar{T}^2k/(N-k) - S_t^2k/N]^{-(k+2)/2}$$

This can be shown as follows, let

$$\ell = [N - t't - k^2\bar{T}^2/(N-k)]/(N-k)$$

$$\varphi = t + k\bar{T}/(N-k)$$

direct computations shows that

$$\frac{D(t)}{D(\bar{t})} = \ell^{-3K/2}(N-k)^{-k} \left| [(N-k)\,\vartheta_{ij} + 1]\,\ell + \varphi_i\,\varphi_j \right|$$

the determinant is that of the matrix

$$(N-k)\,\ell\,(I + [1/(N-k)]u'u + [1/(N-k)\ell\,]\,\varphi'\varphi\,)$$

where u is a row of one u = (11...11), it results from the identity

$$\left| I + X_1'Y_1 + X_2'Y_2 \right| = \begin{vmatrix} 1 + Y_2X_1' & Y_2X_1' \\ Y_1X_2' & 1 + Y_2X_2' \end{vmatrix}$$

that this determinant is

$$(N-k)^k\,\ell^{\,k}[(1 + \frac{\varphi\varphi'}{(N-k)\ell})(1 + \frac{k}{N-k}) - \frac{k^2\bar{\varphi}^2}{(N-k)^2\ell}]$$

$$= (N-k)^{k-2}\,\ell^{\,k-1}\,N^2$$

which gives the result above. Then, g(t)dt is easily obtained as

$$g(t) = h(t)\frac{D(t)}{D(\bar{t})}$$

In case k=1 this reduces to

$$t = \dagger[N(N-1)]^{1/2}/(1-\dagger^2/N-1)^{1/2}$$

$$\frac{D(t)}{D(\dagger)} = [N(N-1)]^{1/2}/(1-\dagger^2/N-1)^{3/2}$$

4.3 Another line of generalization arises from the consideration of distributions admitting a sufficient statistic (scalar or vector). The problem is to find the conditional parameter free distribution of a subsample ξ of size k given the sufficient statistic obtained with a sample X of size N. Nothing very general can be said about this problem without making further assumptions.

In case the distribution of an individual observation belongs to Koopman's family, with one parameter) namely

$$f(X_i)dX_i = \exp[g(\theta)+t(X_i)h(\theta) + \varphi(X_i)]dX_i, h'_\theta \neq 0$$

a change of scale of the parameter in an interval where $h(\theta)$ is monotone gives

$$f(X_i)dX_i = g(\theta)\, e^{t(X_i)\theta+\phi(X_i)}\, dX_i$$

Then, it is well known that the characteristic function of the sufficient statistic $T(X) = \sum_1^N t(X_i)$ is

$$\varphi_{T(X)}(u) = [g(\theta)]^N/[g(\theta+iu)]^N$$

This is straight forward since

$$\int e^{iut(X)} f(X)dX = g(\theta) = \int e^{t(X)[iu+\theta]+\phi(X)} dx = g(\theta)/[g(\theta+iu)]$$

In any case, the probability distribution of T(X) can be obtained by recurrence, using the classical convolution formulas. Under proper conditions of continuity and boundedness,[14] and with obvious notations,

$$p_{n+k}(T_{n+k}) = \int p_k(T_{n+k} - T_k)p_n(T_n)dT_n$$

$$= \int p_n(T_{n+k} - T_k)p_k(T_k)dT_k$$

To obtain the distribution of a subsample ξ of size k given the sufficient statistic T(X) of a sample X of size N, one gets the distribution of $T(\xi_1)\ldots T(\xi_k)$ given T(X) and comes back to the original variable ξ. The characteristic function of $T(\xi_1)\ldots T(\xi_k)$ is $[g(\theta)]^k / \prod_j^k g(\theta + iu_j)$, that of $T_{n-k} =$

$T(X) - T(\xi)$ is $[\varphi(\theta)/\varphi(\theta+iv)]^{N-k}$ that of $T(\xi_1)\ldots T(\xi_k), T_{n-k}$

is $[g(\theta)]^N / [\varphi(\theta+iv)]^{N-k} \prod_j^k g(\theta+iu_j)$, hence the characteristic

function of $T(\xi_1)\ldots T(\xi_k), T(X)$ is

$$[g(\theta)]^N / [g(\theta+iv)]^{N-k} \prod_j^k g[\theta+i(u_j+v)]$$

since $E[e^{u_1 T(\xi_1)+\ldots+u_k T(\xi_k)+vT(X)}]$

$$= E[e^{(u_1+v)T(\xi_1)+\ldots+(u_k+v)T(\xi_k)+vT_{N-k}}]$$

from which one gets the characteristic function of $T(\xi_1)\ldots T(\xi_k)$ given $T(X)$ whose characteristic function is

$$[g(\theta)/g(\theta+iv)]^N$$

The result is

$$\varphi_{T(\xi_1)\ldots T(\xi_k) \mid T(X)}(u_1,\ldots,u_k) = \frac{\int e^{-ivT(X)}[g(\theta+iv)]^{k-N}\prod_j^k (g[\theta+i(u_j+v)])^{-1} dv}{\int e^{-ivT(X)}[g(\theta+iv)]^{-N}\, dv}$$

It is always possible to obtain the probability distribution of ξ given $T(X)$ by writing the joint probability distribution of $T(\xi_1)\ldots T(\xi_k)$ and $T_{N-k} = T(X) - T(\xi)$, making the change of variables $T_{N-k} = T(X) - T(\xi)$ and dividing by the distribution of $T(X)$. To obtain the distribution of $T(\xi)$ given $T(X)$ one writes the joint distribution of $T(\xi)$ and T_{N-k}, makes the change of variables $T_{N-k} = T(X) - T(\xi)$ and divides by the distribution of $T(X)$. If $T = T(X)$ is complete and if $\{\varphi_n(T_n)\}$ is a sequence of unbiased estimates of a function $g(\theta)$ of the parameter θ, then

$$\varphi_N(T) = \int \varphi_k[T(\xi)]\, f(T(\xi) \mid T)\, dT(\xi)$$

i.e., minimum variance unbiased estimates are to be found among the solutions of the integral equation above. Also,

$$\int\ldots\int \psi(\xi)\, f(\xi \mid T)\, d\xi$$

will be an unbiased minimum variance estimate of $E[\psi(\xi)]$ if it exists.

Since the characteristic function $I_A(X_i)$ of a set A is an unbiased estimate of

$$\int_A f(X_i)\,dX_i$$

the best estimate of this measure will be given by

$$\int_A f(\xi_i \mid T)\,d\xi_i$$

and consequently the best estimate of $f(X_i)$ by $f(\xi_i \mid T)$. The discrete case can be treated in a similar way.

4.4 In this paragraph, the straightforward technique mentioned above is applied to several important distributions.

a) Poisson's distribution.

In case

$$P(X_i) = e^{-m} m^{X_i} / X_i!$$

$X_1 + \ldots + X_N = N\overline{X}$ is a sufficient statistic, Poisson distributed and it is complete, one obtains

$$P(\xi \mid \overline{X}) = [1-(k/N)]^{N\overline{X}} [1/(N-k)]^{k\overline{\xi}} \frac{(N\overline{X})!}{\xi_1! \ldots \xi_k! [N\overline{X} - k\overline{\xi}]!} \quad k \neq N$$

on the domain $N\overline{X} - k\,\overline{\xi} > 0$

In case k=N, one obtains the well known result

$$P(\xi \mid \overline{X}) = (N\overline{X})! / \left(\prod_1^k N^{\xi_i} \xi_i!\right)$$

the best estimate of $P(X_i \leq X)$ is

$$P(\xi_i \leq x) = \sum_{1}^{x} [1-(1/N)]^{N\overline{X}} [1/(N-1)]^{\xi_i} \binom{N\overline{X}}{\xi_i} \qquad , x \leq N\overline{X}$$

also

$$P(\overline{\xi} \mid \overline{X}) = (k/N)^{k\overline{\xi}} [1-(k/N)] \binom{N\overline{X}}{k\overline{\xi}} \qquad , N\overline{X} - k\overline{\xi} \geq 0$$

This is a Binomial distribution.

b) Gamma distribution $\Gamma(\alpha, h)$

$$f(X_i)dX_i = [\alpha^h / \Gamma(h)] X_i^{h-1} e^{-\alpha X_i} dX_i,$$

if h is known $\overline{X}$ is sufficient for α, complete, and $N\overline{X}$ is $\Gamma(\alpha, Nh)$ distributed, one obtains

$$f(\xi \mid \overline{X})d\xi = \frac{\Gamma[Nh]}{[\Gamma(h)]^k \Gamma[(N-k)h]} \prod_{1}^{k} (\xi_i / N\overline{X})^{h-1}$$

$$[1-(\overline{\xi}/\overline{X})(k/N)]^{(N-k)h-1} d\xi / (N\overline{X})^k \qquad , \overline{\xi} k \leq N\overline{X}$$

i.e. the $\xi_i / N\overline{X}$ are parameter free

the distribution of $\overline{\xi} \mid \overline{X}$ is

$$f(\overline{\xi} \mid \overline{X})d\overline{\xi} = \frac{1}{B[kh, (N-k)h]} \left(\frac{k}{N}\frac{\overline{\xi}}{\overline{X}}\right)^{kh-1} \left(1-\frac{k}{N}\frac{\overline{\xi}}{\overline{X}}\right)^{(N-k)h-1} d(k\overline{\xi}/N\overline{X})$$

i.e. $k\overline{\xi}/N\overline{X}$ is $B[kh, (N-k)h]$ distributed.

The best estimate of $P(X_i \leq x)$ is, for $x \leq N\overline{X}$,

$$\int_0^x \frac{1}{B[h, (N-1)h]} \left(\frac{\xi_i}{N\overline{X}}\right)^{h-1} \left(1-\frac{\xi_i}{N\overline{X}}\right)^{(N-1)h-1} d\xi_i / N\overline{X} \qquad \xi_i \leq N\overline{X}$$

b') In case h=1, $\Gamma(\alpha,h)$ is the exponential distribution, one obtains,

$$f(\xi \mid \overline{X})d\xi = [\Gamma(N) / \Gamma(N-k)] \cdot \left(1-\frac{\overline{\xi}}{\overline{X}}\frac{k}{N}\right)^{N-k-1} d\xi / (N\overline{X})^k.$$

$$f(\bar{\xi} \mid \bar{X})d\bar{\xi} = [1/B(k,N-k)]\left(\frac{k}{N}\frac{\bar{\xi}}{\bar{X}}\right)^{k-1}\left(1-\frac{k}{N}\frac{\bar{\xi}}{\bar{X}}\right)^{N-k-1} d(k\bar{\xi}/N\bar{X})$$
$$k\bar{\xi} \leq N\bar{X}$$

The best estimate of $P(X_i \leq x)$ is, for $n \leq N\bar{X}$,

$$\int_0^x [1/B(1,N-1)]\,(1-\xi_i/N\bar{X})^{N-2} d(\xi_i/N\bar{X}) \qquad \xi_i \leq N\bar{X}$$

b") The case of the so-called Weibull's distribution

$$f(X_i)dX_i = \alpha P(PX_i)^{\alpha-1} e^{-(PX_i)^{\alpha}} dX_i$$

for known α is not different from that of the exponential distribution except for a change of variable.

c) Binomial distribution

X_i is 0 with probability $1-p$ and 1 with probability p,

$X^* = X_1 + \ldots + X_N$ is sufficient, complete and Binomial distributed

$$P(\xi \mid X^*) = \binom{N-k}{X^* - \Sigma\,\xi_i} / \binom{N}{X^*}$$

with $\xi^* = \xi_1 + \ldots + \xi_k$

$$P(\xi^* \mid X^*) = \binom{k}{\xi^*}\binom{N-k}{X^* - \xi^*} / \binom{N}{X^*}$$

a hypergeometric distribution.

d) Multinomial distribution

$$P(X^1 \ldots X^m) = (N!/X^1! \ldots X^m!)\,p_1^{X^1} \ldots p_m^{X^m}, \quad X^1 + \ldots + X^m = N$$

$(X^1, \ldots, X^m)$ is a complete sufficient statistic X, $\xi = (\xi^1 \ldots \xi^m)$

$P(\xi \mid X) = \binom{X^1}{\xi^1} \cdots \binom{X^m}{\xi^m} / \binom{N}{k} \qquad \xi^i \leq X^i$

e) Rectangular distribution,

$f(X_i)dX_i = dX_i/\theta, \ o \leq X_i \leq \theta, \theta > o$

The observations being ordered $X_{(1)} \leq \cdot \leq X_{(i)} \leq \cdot \leq X_{(N)}$,

$X_{(N)}$ is a sufficient and complete statistic.

One picks a set of k observations $\xi_{(1)} \ldots \xi_{(k)}$.

The probability that

$\xi_{(k)} = X_{(N)}$ is k/N and that $\xi_{(k)} \neq X_{(N)}$ is $[1-(K/N)]$.

if $\xi_{(k)} \neq X_{(N)} \qquad \xi = (\xi_{(1)} \ldots \xi_{(k)})$

$f(\xi \mid X_{(N)})d\xi = k![(N-k)/N]d\xi / X^k_{(N)} \qquad o \leq \xi_{(k)} < X_N$

$f(\xi_{(k)} \mid X_{(N)})d\xi_{(k)}$

$= [(k(n-k))/N](\xi_{(k)}/X_{(N)})^{k-1} \, d\xi_{(k)}/X_{(N)} \qquad o \leq \xi_{(k)} < X_N$

f) Epstein-Sobel's distribution

$f(X_i)dX_i = \alpha e^{-\alpha(X_i - \theta)} dX_i \qquad X_i \geq \theta$

$X_{(1)} \leq \cdots \leq X_{(i)} \leq \cdots \leq X_{(N)}$

then $(X_{(1)}, \bar{X} - X_{(1)})$ is a sufficient and complete statistic.

The probability that $\xi_{(1)} = X_{(1)}$ is k/N and that $\xi_{(1)} \neq X_{(1)}$ is $1-(k/N)$.

The distribution of $(X_{(1)}, \bar{X} - X_{(1)} = Y)$ is

$$\frac{(\alpha N)^N}{\Gamma(N-1)} Y^{N-2} e^{-N\alpha(X_{(1)} + Y - \theta)} dX_{(1)} dY$$

Considering $\xi_{(1)} \neq X_{(1)}$, let X' be the complementary sample of ξ, then

$$X'_{(1)} = X_{(1)}$$

$$Y = \bar{X} - X_{(1)} = \frac{(N-k)\bar{X}' + k\,\bar{\bar{\xi}}}{N} - X_{(1)} = [(N-k)/N]Y' + (k/N)(\bar{\bar{\xi}} - X'_1)$$

i.e. $Y' = [NY/(N-k)] - (k/N)(\bar{\xi} - X_1)$

the usual technique gives

$$f(\xi \mid X_{(1)}, \bar{X}-X_{(1)}) = \frac{k!(N-2)!(N-k)}{(N-k-2)!N}\left[1-(k/N)\left(\frac{\bar{\xi} - X_{(1)}}{\bar{X} - X_{(1)}}\right)\right]^{N-k-2} d\xi / N^k(\bar{X}-X_{(1)})^k$$

in the domain where the quantity between brackets is positive.

$$f(\xi_{(1)}, \bar{\xi} - \xi_{(1)} \mid X_{(1)}, \bar{X} - X_{(1)}) =$$

$$k(k-1)\binom{N-2}{k}[(N-k)/N]\left[\frac{k(\bar{\xi} - \xi_{(1)})}{N(\bar{X} - X_{(1)})}\right]^{k-2}\left[1 - \frac{k(\bar{\xi} - X_{(1)})}{N(\bar{X} - X_{(1)})}\right]^{N-k-2}$$

$$\frac{dk\,\xi_{(1)}}{N(\bar{X}-X_{(1)})} \cdot \frac{dk(\bar{\xi} - \xi_{(1)})}{N(\bar{X} - X_{(1)})}$$

in the same domain.

The first formula is in agreement with a result by R. F. Tate, [15], p. 361, formula (7.8).

[1] W. R. Thompson, "On a Criterion for the Rejection of Observation", A.M.S., 6, (1935), p. 214 ff.

[2] A. G. Laurent, "Distribution d echantillon et de caracteristiques d echantillon guand la population de reference est Laplace Gaussienne de parametres inconnus", Jour. Soc. Stat. Paris, 10-11-12, Oct. 1955, pp. 262-296.

[3] A. G. Laurent,"Generalization of Thompson's Distribution," A.M.S., Dec. 1956, p. 1184 and M.S.U. Research Memorandum, No. 34, Sept. 1956.

[4] A. G. Laurent,"Bombing Problems, a Statistical Approach," Jour. Oper. Res. Soc. Am., Feb. 1957, 5, 1, pp. 75-89 and "Errata", Oper. Res., 6, 2, March 1958, p. 297.

[5] M. D. Springer,"Joint Sampling Distribution of Mean and Standard Deviation. . .," A.M.S., 24, (1953), pp. 118-122.

[6] L. Truska,"The Simultaneous Distributions in Samples of Mean and Standard Deviation and of Mean and Variance," Biometrika, 31, (1940), pp. 256-271.

[7] A. J. Craig,"The Simultaneous Distribution of Mean and Standard Deviation in Small Samples," A.M.S., 3, (1932), pp. 126-140.

[8] P. C. Mahalanobis, R. C. Bose and S. N. Ray,"Normalization of Statistical Variates and the Use of Rectangular Coordinates in the Theory of Sampling Distributions," Sankhya, 3, (1937), pp. 1-40.

[9] P. L. Hsu, Proceedings of the Edinburgh Mathematical Society, 6, Ser. 2, pp. 185-189, (1940).

[10] I. Olkin and S. Roy, "On Multivariate Distribution Theory", A.M.S., 25, 2, pp. 329-339.

[11] W. L. Deemer and I. Olkin, "The Jacobians of Certain Matrix Transformations Useful in Multivariate Analysis", based on lectures of P. L. Hsu, Biometrika, 28, (1951), pp. 345-367.

[12] F. W. Ponting and H. S. A. Potter, " The volume of Orthogonal and Unitary Space", Quart, J. Math., Oxford, 20 (1949), pp. 146-154.

[13] P. Halmos, "Finite Dimensional Vector Spaces", Van Nostrand, N. Y., 1958.

[14] H. Cramer, "Mathematical Methods of Statistics," Princeton, 1946, pp. 188-191.

[15] R. F. Tate, "Unbiased Estimation: Functions of Location and Scale Parameters," A.M.S., 30, 2, pp. 341-366.

Log-Normal Distribution

Estimation Problems

by Andre G. Laurent
Department of Mathematics
Wayne State University

The use of the Log-Normal distribution has been rarely advocated in the literature devoted to industrial life testing and failure theory, though considerable emphasis has been put on that tool in papers dealing with sensitivity data and it has been widely used as a model in the field of biological assays. (Refer, however, to "The Exponential Distribution and Its Role in Life Testing", Technical Report No. 2, May, 1958, by B. Epstein).

A somewhat extensive exposition of the history of the Log-Normal distribution is given in [1].

A broad variety of realistic and "reasonable" hypotheses about the mechanism of failure leads to a Log-Normal distribution for the age of death or the intensity of the stimulus under which failure occurs. Among them, that class of hypotheses which involve random effects that are multiplicative instead of additive and lead to the use of the central limit theorem. For example, if failure is caused by an accumulation of infinitesimal random shocks whose effects are not independent but are sequentially proportional to the already accumulated total effect. This is a well known set up and a plausible model for failure data when an ageing process takes place.

Let X be Log-normal distributed, i.e. Y = Log X is $N(m,\sigma)$ distributed. The moments of X are

$$M_r(X) = e^{r^2\sigma^2/2} e^{rm}$$

whence the expected value and variance of X

$$m_1 = E[X] = e^{\sigma^2/2} e^{m}$$

$$\mu_2 = \mathrm{var}[X] = e^{2m}(e^{\sigma^2}-1)$$

while

$E[Y] = m$, logarithmic mean

$\mathrm{var}[Y] = \sigma^2$ logarithmic variance

the minimum variance unbiased estimates of m and σ^2 being $\bar{Y}$ and $[n/(n-1)]S_Y^2$.

m and σ^2 are measures of central tendency and dispersion in the logarithmic scale, useful corresponding characteristics in the original scale of X are $\mu = e^m$ that is the median of the distribution and $\not{\Sigma}^2 = e^{\sigma^2}$ or $\not{\Sigma} = e^{\sigma^2/2}$.

"Naive" estimates of μ and $\not{\Sigma}^2$ are $e^{\bar{Y}}$ and $e^{S_Y^2}$.

$e^{\bar{Y}} = \exp[\sum_i \frac{\log}{n} X_i] = (\prod_i X_i)^{1/n}$ is the geometric mean of the X_i's

These two estimates are not unbiased;

since $\bar{Y}$ is $N(m,\sigma/\ n)$ $E[e^{\bar{Y}}] = e^m . e^{\sigma^2/2n}$)

and since the generating function of S^2 is $(1-2\sigma^2 t/n)^{-(n-1)/2}$

$$E[e^{S^2}] = (1 - 2\sigma^2/n)^{-(n-1)/2}$$

We want to obtain estimates without bias

1) $e^{-\sigma^2/2n} = \sum_k [(-1)^k/k!][\sigma^{2k}/(2n)^k]$

also

$$E[S_Y^{2k}] = 2^k \frac{\Gamma[k+(n-1)/2]}{\Gamma[(n-1)/2]}(\sigma^{2k}/n^k)$$

i.e. $(n^k/2^k)\dfrac{\Gamma[(n-1)/2]}{\Gamma[k+(n-1)/2]} S_Y^{2k}$ is an unbiased estimate of σ^{2k}

therefore

$$\Gamma[(n-1)/2]\sum_k [(-1)^k/k!]\frac{S_Y^{2k}}{\Gamma[k+(n-1)/2]2^{2k}} \text{ is an}$$

unbiased estimate of $e^{-\sigma^2/2n}$ but this is

$$[2^{(n-3)/2}/S_Y^{(n-3)/2}]\Gamma[(n-1)/2]\, J_{(n-3)/2}(S_Y)$$

where J denotes the Bessel function of first kind of order $(n-3)/2$.

Since $\overline{Y}$ and S_Y are independent variables

$$e^{\overline{Y}}[2^{(n-3)/2}/S_Y^{(n-3)/2}]\Gamma[(n-1)/2]J_{(n-3)/2}(S_Y)$$

is an unbiased estimate of $e^m = \mu$; since $\overline{Y}$ and S_Y are sufficient statistics, it is a minimum variance unbiased estimate of μ.

2) One has

$$e^{h\sigma^2} = \sum_k [h^k/k!]\sigma^{2k}$$

therefore

$$\Gamma[(n-1)/2]\sum_k [h^k/k!][n^k/2^k]\frac{S_Y^{2k}}{\Gamma[k+(n-1)/2]} \text{ is an un-}$$

biased estimate of $e^{h\sigma^2}$, now this is

$$\Gamma[(n-1)/2]\ \frac{2^{(n-3)/2}}{(S_Y\sqrt{2hn})^{(n-3)/2}}\ I_{(n-3)/2}(S_Y\sqrt{2hn})$$

where I denotes the modified Bessel function of first kind of order $(n-3)/2$.

Since S_Y is a sufficient statistic, the minimum variance unbiased estimate of e^{σ^2} and $e^{\sigma^2/2}$ are respectively

$$\Gamma[(n-1)/2]\ \frac{2^{(n-3)/2}}{(S_Y\sqrt{2n})^{(n-3)/2}}\ I_{(n-3)/2}(S_Y\sqrt{2n}) \quad \text{and}$$

$$\Gamma[(n-1)/2]\ \frac{2^{(n-3)/2}}{(S_Y\sqrt{n})^{(n-3)/2}}\ I_{(n-3)/2}(S_Y\sqrt{n})$$

As a by-product, one will obtain the minimum variance unbiased estimate of any moment $M_r(x)$ of X.

Since

$E[e^{r\bar{Y}}] = e^{rm}.e^{r^2\sigma^2/2n}$

depends only on $\bar{Y}$, and we have an unbiased estimate of $e^{h\sigma^2}$ that depends only on S_Y, making $h = (n-1)r^2/2n$, we get a minimum variance unbiased estimate of M_r as

$$e^{r\bar{Y}}.\Gamma[(n-1)/2]\ \frac{2^{(n-3)/2}}{(rS_Y\sqrt{n-1})^{(n-3)/2}}\ I_{(n-3)/2}(rS_Y\sqrt{n-1})$$

References

[1] J. C. Aitchinson and J. A. C. Brown, "The Log-Normal Distribution," Cambridge, 1957.

WAYNE STATE UNIVERSITY
Technical Report Distribution List
Contract Nonr-2575(00)

Naval Activities

Head, Statistics Branch 2
Office of Naval Research
Washington 25, D.C.

Commanding Office 2
Office of Naval Research Branch Office
346 Broadway
New York 13, New York

Commanding Officer 2
Office of Naval Research Branch Office
Navy No. 100
Fleet Post Office
New York, New York

Commanding Officer 1
Office of Naval Research
1000 Geary Street
San Francisco 9, California

Technical Information Center 6
Naval Research Laboratory
Washington 25, D.C.

Chief of Naval Materiel 1
Code M533, Room 2236
Main Navy Building
Department of the Navy
Washington 25, D.C.

Dr. Craig A. Magwire 1
Department of Math. and Mech.
U. S. Naval Postgraduate School
Monterey, California

Dr. Mitchell L. Cotton 1
Department of Electronics
U. S. Naval Postgraduate School
Monterey, California

Chief, Bureau of Ordnance (Ad3) 1
Department of the Navy
Washington 25, D.C.

Chief, Bureau of Ordnance 1
Department of the Navy
Washington 25, D.C.
Attn: A. Rothstein Code ReUg

Chief, Bureau of Aeronautics 1
Department of the Navy
Washington 25, D.C.
Attn: Quality Control Div.

Commander 1
U.S. Naval Air Missile Test Ctr.
Point Mugu, California
Attn: Chief Scientist

Commander 1
U. S. Naval Air Development Ctr.
Johnsville, Pennsylvania

Mr. H. R. Thoman 1
Bureau of Aeronautics, AV-4422
Department of the Navy
Room 2W96, W Building
Washington 25, D.C.

Mr. Francis A. Thompson 1
Bureau of Aeronautics, AV-3102
Department of the Navy
Room 1W63, W Building
Washington 25, D.C.

Chief, Bureau of Yards and Docks [illegible]
Department of the Navy
Washington 25, D.C.
Attn: Mr. [illegible]. Wolman, Code [illegible]

Quality Control Division (Qcc) 1
Bureau of Ordnance
Department of the Navy
Washington 25, D.C.

Statistics Branch 1
Qc Division Qc5
Bureau of Ordnance
Department of the Navy
Washington 25, D.C.

Mr. R. E. Wiley 1
Materiels Branch
Office of Naval Research
Washington 25, D.C.

Mr. Masao Yoshitsu 1
Naval Inspector of Ordnance
Naval Gun Factory
Washington 25, D.C.

Miss Besse B. Day 1
Bureau of Ships, Code 310B
Department of the Navy
Washington 25, D.C.

Mr. A. Lieberman 1
Bureau of Ships, Code 3730
Department of the Navy
Washington 25, D.C.

Mr. A. S. Marthens 1
Bureau of Ships, Code 373A
Department of the Navy
Washington 25, D.C.

Mr. P. Brown 1
Bureau of Ships, Code 363B
Department of the Navy
Washington 25, D.C.

Mr. H. Weingarten 1
Bureau of Ships, Code 280
Department of the Navy
Washington 25, D. C.

Mr. F. R. DelPriore 1
U. S. Naval Engineering Experiment Station
Annapolis, Maryland

Captain B. L. Lubelsky, USN 1
Quality Evaluation Laboratory
U. S. Naval Ammunition Depot
Crane, Indiana

Commanding Officer 1
U. S. Navy Mine Defense Laboratory
Panama City, Florida
Attn: Mr. J. Boyd

Commander 1
Materiel Laboratory
New York Naval Shipyard
Naval Base
Brooklyn 1, New York
Attn: A. Walner

Commanding Officer 1
U. S. Naval Radiological Defense Laboratory
San Francisco, California
Attn: Miss M. Sandomire

Dr. Julius Lieblein 1
Applied Mathematics Lab. Code 820
David Taylor Model Basin
Washington 7, D.C.

Mr. E. J. Nucci 1
Bureau of Ships, Code 819
Department of the Navy
Washington 25, D.C.

Office, Secretary of Defense

Inspection and Quality Control Division 1
Office, Asst. Secretary of Defense (S and L)
Washington 25, D.C.
Attn: Mr. Irving B. Altman

Inspection and Quality Control Division 1
Office, Asst. Secretary of Defense (S and L)
Washington 25, D.C.
Attn: Mr. John J. Riordan

Office, Asst. Secretary of Defense (R and E) 1
Room 3D984, The Pentagon
Washington 25, D.C.
Attn: Mr. R. H. DeWitt

Office of Guided Missiles 1
Office, Asst. Secretary of Defense (R and E)
Washington 25, D.C.
Attn: Mr. Carlton M. Beyer

Technical Library 1
Office, Asst. Secretary of Defense (R and E)
Room 3E1065, the Pentagon
Washington 25, D. C.

Army Activities

Mr. Silas Williams, Jr. 1
Standards Branch
Procurement Division
DCS Logistics, U.S. Army
Washington 25, D.C.

Dr. L. S. Gephart 1
Office of Ordnance Research
Box CM, Duke Station
Durham, North Carolina

Air Force Activities

LTCOL W. C. Marcuis 1
Air Research Development Command
Baltimore, Maryland

Mr. R. Riedenbender 1
Hq., Air Materiel Command (MCQFF)
Wright-Patterson Air Force Base
Ohio

Dr. J. A. Greenwood
Hq., U. S. Air Force, AFCIN-3BX2
Washington 25, D.C.

Miscellaneous Government Agencies

ASTIA Documents Service Center 5
Knott Bldg.
Dayton 2, Ohio

Office of Technical Services 1
Department of Commerce
Washington 25, D.C.

Mr. Leon Gilford, ADSS 1
U. S. Census Bureau
Washington 25, D.C.

Dr. Joseph Daly 1
U. S. Census Bureau
Washington 25, D.C.

Dr. Marvin Zelen 1
Statistical Engn'g Laboratory
National Bureau of Standards
Washington 25, D.C.

Business Concerns

Mr. I. Belson 1
Mine Safety Appliances Co.
Gallery, Pennsylvania

Mr. John K. Borkman 1
Whippany Laboratory
Bell Telephone Laboratories
Whippany, New Jersey

Mr. W. H. Clatworthy 1
Bettis Plant
Westinghouse Electric Corp.
Box 1468
Pittsburgh 30, Pennsylvania

Mr. H. F. Dodge 1
Bell Telephone Laboratories, Inc.
463 West Street
New York 14, New York

Mr. S. Weiner 1
ARINC Research Corporation
1700 K Street, N.W.
Washington 6, D.C.

Dr. G. Kaskey 1
Remington Rand UNIVAC Div. of
Sperry Rand Corporation
1900 West Allegheny Avenue
Philadelphia, Pennsylvania

Dr. John F. Hofmann 1
Systems Laboratories Corp.
15016 Bentura Blvd.
Sherman Oaks, California

Mr. J. C. Duryea 1
Senior Components Engineer
Sperry Gyroscope Company
Great Neck, New York

Universities

Professor Cyrus Derman 1
Depart. of Industrial Engn'g
Columbia University
New York 27, New York

Professor Acheson J. Duncan 1
Dept. of Industrial Engn'g
The Johns Hopkins University
Baltimore 18, Maryland

Professor Eugene L. Grant 1
Civil Engineering Department
Stanford University
Stanford, California

Dr. Eugene Lukacs 1
Department of Mathematics
Catholic University
Washington 17, D.C.

Professor E. G. Olds 1
Department of Mathematics
Carnegie Institute of Technology
Pittsburgh 13, Pennsylvania

Dr. Max A. Woodbury 1
Department of Mathematics
College of Engineering
New York University
New York 53, New York

Professor Ralph A. Bradley 1
Department of Mathematics
Virginia Polytechnic Institute
Blacksburg, Virginia

Professor Ellis Ott 1
Department of Mathematics
Rutgers University
New Brunswick, New Jersey

Part II - Foreign Addresses

Professor Maurice H. Belz
University of Melbourne
Carlton N. 3
Victoria, AUSTRALIA

Professor Tosio Kitagawa
Mathematical Institute
Faculty of Science
Kyushu University
Fukuoka, JAPAN

Mr. Cesareo Villegas
Instituto de Matematica y Estadistica
Av. J. Herrera y Reissig 565
Montevideo, URUGUAY

Mr. F. F. Wade, Statistician
Aluminum Company of Canada, Ltd.
Kingston, Ontario, CANADA

Professor D. A. S. Fraser
Department of Mathematics
University of Toronto
Toronto 5, CANADA